Heuristic Derivations of Black Hole Properties Using Dimensional Analysis

’Kale Oyedeji* and Ronald E. Mickens**

Department of Physics*
Morehouse College
Atlanta, GA 30314-3773
USA

and

Department of Physics**
Clark Atlanta University
Atlanta, GA 30314
USA
Gravitational Interaction Between Two Bodies

- Let $m_1 \ll m_2 = M$
- Let $V(r) = \text{gravitational interactional potential energy.}$

Therefore

\[
V(r) = -\frac{G M m_1}{r} = \left(\frac{G M / c^2}{r}\right) m_1 c^2
\]

\[
V(r) = -\left(\frac{R_{BH}}{r}\right) m_1 c^2.
\]

- $R_{BH}$ is defined to be the ”black hole” radius of the body having mass $M$.

- $c = \text{speed of light in a vacuum.}$
$V(r) = -\left(\frac{R_{BH}}{r}\right) (m_1 c^2)$

- The factor $m_1 c^2$ is the rest energy of the smaller mass $m_1$.
- The factor $\left(\frac{R_{BH}}{r}\right)$ is a scaling factor and is dimensionless.

It represents the ratio of the BH radius to the distance of $m_1$ from the center of massive body $M = M_{BH}$.

- Let $R_0$ be the physical radius of the massive body $M$. Then the weakness of the gravitational interaction is a consequence of

$$R_0 \gg R_{BH}.$$ 

- **Note:** The standard (Schwartzchild) radius of a BH is

$$R_{BH}^{(S)} = 2R_{BH}.$$
Results for Earth and Sun

Sun: \[ R_0 = (6.96)10^5 \text{Km}, \quad R_{BH} = 1.5\text{Km} \]
Earth: \[ R_0 = (6.37)10^3 \text{Km}, \quad R_{BH} = (4.5)10^{-6}\text{Km} \]

Geometric Properties of BH’s

- **Surface Area**
  
  Assuming a spherical shape gives

\[
SA_{BH} = 4\pi (R_{BH})^2 = \left(\frac{4\pi G^2}{c^4}\right)(M_{BH})^2
\]

- **Volume**

\[
V_{BH} = \left(\frac{4}{3}\right)4\pi (R_{BH})^3 = \left(\frac{4}{3}\right)\left(\frac{4\pi G^3}{c^6}\right)(M_{BH})^3
\]

- **Density**

\[
\rho_{BH} = \frac{\text{mass}}{\text{volume}} = \left(\frac{3}{4}\right)\left(\frac{c^6}{4\pi G^3}\right)\left[\frac{1}{(M_{BH})^2}\right]
\]
This thermal radiation is often called ”Hawking radiation”. 


”Dimensional analysis is a powerful method to understand the relation between physical quantities in a given problem.

The basic idea is, all scientifically interesting results are expressed in terms of dimensionless quantities, independent of the system units you are using. There are two consequences of this idea: first, one can often guess a reasonable form of an answer just by thinking of the dimensions in complex physical situations, and test the answer by experiments or more developed theories. Second, dimensional analysis is routinely used to check the plausibility of derived equations or computations.”
Parameters/Physical Constants

- $c =$ speed of light
- $G =$ gravitational constant
- $M_{BH} =$ mass of BH
- $K_B =$ Boltzmann’s constant
- $\hbar =$ Normalized Plank’s constant
- $T_{BH} =$ effective temperature of BH

Assuming the following relationships hold

$$K_B T_{BH} = \lambda C^a (\hbar)^b G^c (M_{BH})^d.$$  

$\lambda =$ real number of order one.

Since the left-side has units of energy, then expressing the right-side in terms of mass ($M$), length ($L$), and time ($T$), gives

$$M L^2 / T^2 = L^{(a+2b+3c)} M^{(b-c+d)} T^{(-a-b-2c)}.$$
Therefore, comparing the exponential on both sides gives

\[
\begin{align*}
  a + 2b + 3c &= 2 \\
  b - c + d &= 1 \\
  a + b + 2c &= 2
\end{align*}
\]

Since there are four variables \((a,b,c,d)\) and three equations, let’s express \((a,b,d)\) in terms of \(c\), to give

\[
a = 2-c, \quad b = -c, \quad d = 1+2c.
\]

Therefore

\[
K_{BH}T_{BH} = \lambda C^{(2-c)}\hbar^{(-c)}G^c M^{(1+2c)}.
\]

Comment

Since overwhelming evidence exists for BH’s and no detectable BH radiation exists, then

\[
1 + 2c < 0 \quad \implies \quad c < -\left(\frac{1}{2}\right).
\]
Replacing \( c = -f \), gives \( f > \frac{1}{2} \) and

\[
K_{BH}T_{BH} = \lambda C^{(2+f)}(\hbar)^{f}/(G)^{f}(M_{BH})^{(2f-1)}
\]

A rigorous calculation gives

\[
K_{BH}T_{BH} = \left(\frac{1}{8\pi}\right) \left(\frac{c^3\hbar}{GM_{BH}}\right),
\]

which implies

\[
f = -C = 1, \quad \lambda = \left(\frac{1}{8\pi}\right) \approx 0.04
\]
Life-Time, $t_{BH}$

- $t_{BH} = \lambda_1 C^a \hbar^b G^c M_{BH}^d$
- Written in terms of $b$, we have
  $$a + 3c = -2b, \quad c - d = b, \quad a + 2c = -(1 + b)$$

  and
  $$a = b - 3, \quad c = 1 - b, \quad d = 1 - 2b.$$ 

- Therefore
  $$t_{BH} = \lambda_1 C^{(b-3)}(\hbar)^b (G)^{(1-b)} (M_{BH})^{(1-2b)}$$

  Since $t_{BH} \to \infty$, as $\hbar \to 0$, then $b < 0$. Thus, for $g = -b > 0$, we have
  $$t_{BH} = \lambda_1 \left[ \frac{(M_{BH})^{(1+g)} (G)^{(1+g)} (\hbar)^g C^{(3+g)}}{\hbar g C^{(3+g)}} \right]$$

  A detailed calculation gives $g = 1$. 

Summary/Conclusion

- We have derived (essentially) all the fundamental properties of a standard black-hole using only the method of dimensional analysis.
- All calculated properties generally agree with the detailed exact analysis (up to an overall multiplicative real constant).
- To the best of our knowledge the derivations of the black hole temperature and life time have not appeared in published form.
- The methodology of this presentation could be used to introduce advanced undergraduates to the physics of black-holes.

Thank You!!!