Conduit Bound Sound Propagation Separation Model

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Fundamental concept

Sound Propagating with the flow will propagate faster than the speed of sound
Speed Left to Right = \( c + v = V(LR) \)

Sound Propagating against the flow will propagate slower than the speed of sound
Speed Right to Left = \( c - v = V(RL) \)

Note:
\( c = \text{Speed of Sound in the fluid} \)
\( v = \text{flow velocity of fluid} \)
Fundamental concept

If we have a method of measuring $V(LR) = c + v$ and $V(RL) = c - v$ we can simply determine both the flow velocity of the fluid and the speed of sound in the fluid

$$c = \frac{(V(LR) + V(RL))}{2}$$

$$v = \frac{(V(LR) - V(RL))}{2}$$

Note:
$c =$ Speed of Sound in the fluid
$v =$ flow velocity of fluid
This simple concept has proven to be not so simple in practice

- Whole texts have been written on the subject (e.g. Beck and Plaskowski 1987)
- The methods were not successful because of the super position principle of waves that caused interference from unwanted sound (i.e. noise)
- This noise could not be removed, hence it had to be measured.
- The noise is no longer a problem—It’s the Solution!
The solution is to listen to the noise already present in the pipe with multiple transducers

- We can listen to the sound through the wall of the pipe, and do not need to cut into the pipe.
- A two dimensional Fast Fourier Transform (2d-FFT) will reveal the necessary information
To a first approximation the Left-to-Right wave can be separated from the Right-to-Left waves in the 2d-FFT

$$\Psi_{LR}(z, t) = Ne^{2\pi i (ft - k\tau)}$$

$$\Psi_{RL}(z, t) = Ne^{2\pi i (ft + k\tau)}$$

The slopes of these lines will equal $V(LR)$ and $V(RL)$

$$c = \frac{(V(LR) + V(RL))}{2}$$

$$v = \frac{(V(LR) - V(RL))}{2}$$
Real data reveals these lines, but a lot more!
The diagonal lines are clearly present, and were explained in U.S. patent US2006/0201430 A1

But, what about these other curves above the diagonal lines?
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To model this wave we must return to the fundamental wave equation

\[ \nabla^2 \Psi = \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} \]

Given that the wave is confined to a cylinder the natural approach is to represent this equation in cylindrical coordinates

\[ \frac{1}{r} \left( \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \Psi \right) + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} \]
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If we now assume a separable solution where $R(r)$ represents the bound radial component of the wave, $\Phi(\phi)$ represents the angular rotation of the wave in the cylinder and $P(z,t)$ represents the unbound propagating wave

\[
\Psi(r,z,t) = R(r)\Phi(\phi)P(z,t)
\]

The wave equation now reduces to the following

\[
\frac{r}{R(r)}\left(\frac{\partial}{\partial r}r\frac{\partial}{\partial r}R(r)\right) + \frac{r^2}{P(z,t)}\left(\frac{\partial^2 P(z,t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 P(z,t)}{\partial t^2}\right)
\]

\[
= - \frac{1}{\Phi(\phi)} \frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = m^2
\]

The solution for the angular wave is now found to be the following

Where $m=0, \pm 1, \pm 2...$

\[
\Phi(\phi) = e^{im\phi}
\]
We now introduce a new constant $A$ where $A$ has the form below, and rearrange to further separate the variables

$$\frac{1}{rR(r)} \left( \frac{\partial}{\partial r} r \frac{\partial}{\partial r} R(r) \right) + \frac{1}{P(z, t)} \left( \frac{\partial^2 P(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 P(z, t)}{\partial t^2} \right) = \frac{m^2}{r^2} = A^2$$

This reduces to the traditional form of the $m$ order cylindrical Bessel function of the first kind

$$r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} + (A^2 r^2 - m^2) R = 0$$

Where $R(r) = J_m(Ar)$
We are now left to consider the unbound portion of the wave that can propagate from Left to Right and from Right to Left.

\[
\frac{1}{P(z, t)} \left( \frac{1}{c^2} \frac{\partial^2 P(z, t)}{\partial t^2} - \frac{\partial^2 P(z, t)}{\partial z^2} \right) = -A^2
\]

This equation has two fundamental solutions.

For both equations we serendipitously have the same result below when substituted into the above equation.

\[
P_{RL}(z, t) = e^{2\pi i (ft + kz)} \text{ for Right to Left}
\]

\[
P_{LR}(z, t) = e^{2\pi i (ft - kz)} \text{ for Left to Right}
\]

\[
(2\pi)^2 \left( \frac{f^2}{c^2} - k^2 \right) = A^2
\]
Given that the velocity of the wave is defined as \( (\mathbf{v} \cdot \nabla \psi) \), and that the radial velocity at the wall must be zero we have the following boundary condition.

\[
\nu_r = -\frac{\partial \psi}{\partial r} = -\frac{\partial J_m(Ar)}{\partial r} = 0
\]

These roots of the first derivative of the Bessel function can only be found numerically.

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This now reduces to a solution that can now model the off-diagonal curves

\[
\left( \frac{2\pi}{\sqrt{\frac{f^2}{c^2} - k^2}} \right) r_w = \text{Root}(m,j)
\]

Note: Root=0.0 yields \( f/k = c \) for the diagonal.
As outlined in U.S. Patent No. 9,441,993 (McGill) these curves can be explained by better wave equations.

\[
\Psi_{LR}(r, z, t) = NJ_m \left[ \left( 2\pi \sqrt{\frac{f^2}{c^2} - k^2} \right) r \right] e^{2\pi i (ft - kz)}
\]

\[
\Psi_{RL}(r, z, t) = NJ_m \left[ \left( 2\pi \sqrt{\frac{f^2}{c^2} - k^2} \right) r \right] e^{2\pi i (ft + kz)}
\]
Thank you

• SESAPS 2016
• Georgia College & State University